Network SIR Model - Meanfield

- Erdös-Renyi graph
- synchronous timestep
- Disease states: S - susceptible
  I - infectious
  R - recovered
- Processes:
  - transmission probability p_i
    (probability of transmitting at next timestep)
  - recovery: p_r

We could write the network model explicitly:

$$x_{i,t+1} = f(x_{it}, \underbrace{x_{j,t} \in N(x_i)}_{\text{neighboring nodes}})$$

v/ high dimensional!
Instead, lets make:

Mean field model

- reduces dimension

Take advantage of the fact that nodes are connected randomly.

Let N be total number of nodes

Define variables:

$$s = \frac{S}{N}$$ = fraction of nodes that are susc.

number of nodes susc.

$$i = \frac{I}{N}$$ = fraction of nodes that are infectious

$$r = \frac{R}{N}$$ = fraction of nodes that are recovered

note $r = 1 - s - i$ since the fractions sum to 1 (population is fixed)

Probability that a susceptible node j is infected by node k is:

$$\boxed{\underbrace{p_e}_{\text{probability that a j-k edge exists}} \times \underbrace{i}_{\text{probability that node k is infected}} \times \underbrace{p_i}_{\text{probability of transmitting}}}$$

Then $(1 - p_e i p_i)^{N-1}$ is the probability that node j remains susceptible at the next time step.

Probability of having an $S \to S$ transition at the next time step, i.e. the fraction of nodes that go $S \to S$ is:

$$\underbrace{s}_{\substack{\text{fraction}\\ \text{of } S}}\underbrace{(1 - p_e i p_i)^{N-1}}_{\substack{\text{probability that}\\ \text{they stay } S}}$$

We can write out our transition probabilities as:
(note we dropped the t subscripts)

Current state	Next state	Probability of transition
Susc	Susc	$s(1 - p_e i p_i)^{N-1}$
Susc	Inf	$s(1 - (1 - p_e i p_i)^{N-1})$
Inf	Inf	$i(1 - p_r)$

Inf	Rec.	i pr
Rec	Rec	r . 1

Write out the difference equations:

$$s_{t+1} = s_t (1 - pe\, i_t\, pi)^{N-1}$$

or we could write:

current − loss

$$s_{t+1} = s_t - \text{transitions to } i$$

$$= s_t - s_t (1 - (1 - pe\, i_t\, p_i)^{N-1})$$

$$= s_t (1 - (1 - (1 - pe\, i_t\, p_i)^{N-1}))$$

$$= s_t (1 - pe\, i_t\, pi)^{N-1})$$

i_{t+1} = incoming from s + stay inf. (don't recover)

$$= s_t(1 - (1 - pe\, i_t\, pi)^{N-1}) + i_t (1 - pr)$$

$= i_t +$ new transmissions

$-$ recoveries

$$r_{t+1} = i\,p_r + r_t$$

$$= 1 - s_{t+1} - i_{t+1}$$

Pulling this all together:

$$s_{t+1} = s_t (1 - p_e i_t p_i)^{N-1}$$

$$i_{t+1} = s_t(1 - (1 - p_e i_t p_i)^{N-1}) + i_t (1 - p_r)$$

$$r_{t+1} = r_t + i_t p_r$$

Binomial Approximation

$$(1+x)^p \quad (\underbrace{x<1 \quad xp<<1}_{\text{small}})$$

$$\approx 1 \times xp$$

Rewriting with the binomial approx (assuming that $\underbrace{p_e i_t p_i}_{\text{small}}$ is small and $\times N-1$ also):

$$(1-p_e i_t p_i)^{N-1} \approx 1-(N-1)p_e i_t p_i$$

$$s_{t+1} = s_t(1-p_e i_t p_i)^{N-1}$$

$$= s_t(1-(N-1)p_e i_t p_i)$$

$$= s_t - \underbrace{\overset{\text{constants}}{((N-1)p_e p_i)}}_{b} s_t i_t$$

$$s_{t+1} = s_t - b s_t i_t$$

$$i_{t+1} = s_t(1-(1-p_e i_t p_i)^{N-1}) + i_t(1-p_r)$$

$$\approx s_t(1-(1-(N-1)p_e i_t p_i)) + i_t(1-p_r)$$

$$= \underbrace{(N-1)p_e p_i}_{b} s_t i_t + i_t(1-p_r)$$

$$= b s_t i_t + i_t - p_r i_t$$

$$s_{t+1} = s_t - b s_t i_t$$

$$i_{t+1} = i_t + b s_t i_t - p_r i_t$$

$$r_{t+1} = r_t + p_r i_t$$

This looks like an SIR model—it is!

Also looks like a discretization of the standard ODE version:

$$\frac{dS}{dt} = -\beta S I$$

$$\frac{dI}{dt} = \beta S I - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

a different model:

$$S \to E \to I \to R$$

$$s_{t+1} = s_t - b s_t i_t$$

$$e_{t+1} = b s_t i_t - p_e e$$

$$i_{t+1} = p_e e - p_r i$$ etc.

SIR Equilibria

How can the mean field model help us?

solve for where:

$$s_{t+1} = s_t$$
$$i_{t+1} = i_t$$
$$r_{t+1} = r_t$$

this is an equilibrium point for our model

This gives us:

$$i_t = 0$$
$$s_t + r_t = 1$$

any set of values for s, r that fit this condition will work.

SIR model can't have long term

eventually the disease must die out.

$i \neq 0$ at a steady state

SIS model Equilibria

$S \rightleftarrows I$

$$\left.\begin{aligned} s_{t+1} &= s_t - b s_t i_t + pr\, i_t \\ i_{t+1} &= i_t + b s_t i_t - pr\, i_t \end{aligned}\right\}$$ mean field for ER network w/ SIS dynamics

note b/c fixed pop. size:

$s_t + i_t = 1 \qquad s_t = 1 - i_t$

We can write the model as 1 equation:

$$i_{t+1} = i_t + b(1-i_t)\, i_t - pr\, i_t$$

Solve for equilibria by setting $i_{t+1} = i_t$:

$$0 = b s_t i_t - pr\, i_t$$

$$0 = (\underbrace{\boxed{b s_t - pr}}_{=0})\, i_t \longrightarrow s = pr/b$$

Two possible equilibria:

<u>Disease free EQ.</u>

$i_t = 0$

$s_t = 1$

<u>Endemic Eq.</u>

$s_t = pr/b$

$i_t = 1 - pr/b$

We can look at stability by noting that

$$\underbrace{i_{t+1} - i_t}_{\substack{\text{if } + \text{ then } i \text{ will grow} \\ \text{if } - \text{ then } i \text{ will decline}}} = \text{change in } i$$

$$i_{t+1} - i_t = \underbrace{(bs_t - pr)}_{\substack{\text{controls whether} \\ \text{the RHS is } + \text{ or } -}} i_t$$

If we start near the DFE then if

$bs_t - pr > 0$ then the epidemic will grow

If $bs_t - pr < 0$ it will decline

at DFE $i_t = 0$ $s_t = 1$

so near DFE:

$\underbrace{bs_t}_{\approx 1} - pr \approx b - pr > 0$ vs < 0

that will control growth of disease

$b > pr$

$R_0 = \boxed{\dfrac{b}{pr}} > 1$

R_0

$R_0 > 1$ the disease $\uparrow$

$R_0 < 1$ the disease $\downarrow$

$R_0 = b \cdot \dfrac{1}{pr}$

Recall -

mean

$$b = \underbrace{\underbrace{(N-1)\,p_e}_{\text{average degree}}\;\underbrace{p_i}_{\text{prob of transmission}}}_{\text{expected number of cases 1 person generates in 1 time step}}$$

$\frac{1}{p_r}$ = approximate number of time steps we expect to stay infected.

Why? (comes from basic probability
will post note)

But anyhow –

Together $R_0 = \frac{b}{p_r}$ = total cases generated by 1 infected person over the time

... infections.

Extra note on why 1/pr = expected number of steps until recovery (we didn't have time in class for this so I figured I'd add it on)

* take an infected node. probability of recovery @ next time = pr

memoryless, so we can write:

$$E[X] = 1 + \underbrace{pr \times 0}_{\text{stop if recovered}} + (1-pr)\,E[X]$$

time to recovery

I have to take one step.

keep going if didn't.

expected value next time is same as now since memoryless

Solve for E[X]:

$$E[X]\left(1-(1-pr)\right) = 1$$

$$E[X] = 1/pr$$